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THE

AMERICAN

MATHEMATICAL MONTHLY

VOLUME XXII FEBRUARY, 1915 NUMBER 2

THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

II.

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B. ARISTOTLE'S EXPOSITION AND CRITICISM.

The purposes of Zeno's arguments, as set forth by Cousin, Grote, and P. Tannery differed from the purpose as it was understood by Plato, Aristotle, and the later Greek writers. The three modern interpreters gave to the arguments settings in the history of Greek thought which exhibit them as flawless in logical rigor. On the other hand, Aristotle and subsequent Greek writers interpreted the arguments as fallacies and expended their mental acumen in attempts to point out the real nature of the fallacies. Except for the recent studies of Cousin, Grote, and Tannery, the history of Zeno's arguments against motion has been for over 2,000 years the history of attempts to explain Zeno's "fallacies." Aristotle acknowledged the great difficulty in exposing their hidden source of logical error. The sixth book of his *Physics* is devoted to the exposition of the subtle notions of continuity and infinity. In fact, the entire *Physics*, of the size of about 225 ordinary modern pages of print, gives all of its eight books to discussions of the notions of motion, divisibility, continuity, infinity, and the vacuum. As one reads the *Physics* one is impressed by the fact that the great Stagirite did not hesitate to restate his arguments repeatedly. For twenty centuries Aristotle's criticisms of Zeno have been the starting points of philosophic discussion; for twenty centuries Zeno's fallacies have puzzled many of the best minds. Like the problems of the trisection of an angle, of the squaring of the circle, and of the duplication of the cube, Zeno's fallacies have challenged some of the best brains; but, like the former problems, have finally been forced to surrender their secrets and to enter the group of "problems of the past."

Before entering upon the study of Aristotle we must refer to the atomistic theory advanced by Leucippus (about 500 B. C.) and developed further by Democritus (about 460 to about 357 B. C.) and others. They conceived magnitudes as composed of indivisible elements in finite number. This view must have gained some ascendancy among a few early mathematicians. Democritus himself was not without mathematical ability. From a notice of Plutarch, Democritus raised the following question:[1]

> "If a cone were cut by a plane parallel to its base,[2] what must we think of the surfaces of the sections, that they are equal or unequal? For if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal the sections will be equal, and the cone will appear to have the property of a cylinder, viz., to be composed of equal, and not unequal, circles, which is very absurd."

This paradox of Democritus places difficulties in the way of accepting the notion of an infinitesimal and thereby indirectly favors the idea of divisibility in only a finite number of parts.

In Antiphon's attempt to square the circle it is assumed that straight and curved lines are ultimately reducible to the same indivisible elements. Antiphon, according to the testimony of Simplicius and Philoponus, inscribed in a circle a square and, by arc-bisection, obtained regular polygons of 8, 16, 32 sides, and so on. He assumed that a polygon could be reached that coincides with the circle. Simplicius observes:[3]

> "The conclusion here is manifestly contrary to geometrical principles, not, as Alexander maintains, because the geometer supposes as a principle that a circle can touch a straight line in one point only, and Antiphon sets this aside; for the geometer does not suppose this, but proves it. It would be better to say that it is a principle that a straight line cannot coincide with a circumference, for one without meets the circle in one point only, one within in two points, and not more, and the meeting takes place in single points. Yet, by continually bisecting the space between the chord and the arc, it will never be exhausted, nor shall we ever reach the circumference of the circle, even though the cutting should be continued *ad infinitum:* if we did, a geometrical principle would be set aside, which lays down that magnitudes are divisible *ad infinitum*."[4]

Aristotle's idea of continuity differs from the idea of continuity as developed by Georg Cantor and his followers. Aristotle's is a sensuous, physical continuum, in which there is an intimate bond between its elements; Cantor's is a collection of elements, arranged in order, infinite in number, but externally to each other; it is purely abstract and transcends the power of the imagination to grasp it. As Aristotle's *Physics* is not available in English translation, we shall translate from it rather freely. Says Aristotle:[5]

> "If things continuous, things touching one another, and things successive are as described above, namely that continuous things are those whose extreme limits are one [united], things

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[1] Plutarch, *de Comm. Not.*, Vol. IV, p. 1321, ed. Didot. Quoted from Allman, *Greek Geometry*, 1889, p. 81.

[2] This passage obviously means, that the cutting plane is infinitely near to the base of the cone.

[3] *Simplicii comment. in octo Aristotelis physicae auscultationis libros.* Venetiis, 1526, § 80. We quote the translation given in G. J. Allman's *Greek Geometry*, 1889, p. 66.

[4] According to another commentator, Themistius (317–387 A. D.), Antiphon began his process of squaring the circle, not by inscribing a square, but by inscribing an equilateral triangle.

[5] Aristotle's *Acht Bücher der Physik.* Griechisch und Deutsch von Carl Prantl, Leipzig, 1854, p. 279, Buch VI, § 1.

touching are those whose extreme limits have the same position, things successive are those between which there lies nothing that is like them, then it is impossible that a thing continuous consist of indivisible things, as for instance, that a line be made up of points, the line being taken as continuous, and a point as indivisible."

This passage is made clearer by a quotation or two from the *Physics*, Bk. V, § 3:

"Continuous are those things which in their union become one in nature; and just as that which holds them in continuity together becomes one, so the whole will be one, as for instance by means of a nail, or glue, or an adherence or an accretion."

"Things continuous must necessarily touch; on the other hand, things touching are not necessarily continuous, for the extreme ends, even though locally together, are not necessarily one and the same."

Returning to the *Physics*, Bk. VI, § 1, we reproduce the chief part of Aristotle's argument which leads him to the conclusion that a line is not made up of points:

> "For neither are the extreme ends of points one [united], for the reason that, of an indivisible, the one cannot be an extreme and the other another part, nor are the extreme ends locally together, since the indivisible can have no extreme."

According to Aristotle, things continuous are always divisible into parts that are continuous. In the same way, time is not made up of parts considered as indivisible *Nows*.[1] We quote from *Physics*, Bk. VI, § 2:

> "As each magnitude is divisible into magnitudes (for it has just been proved that the continuous cannot consist of the indivisible, and every magnitude is continuous), the faster of two must necessarily in the same time be moved through a larger distance and in a less time through an equal distance, and also in a less time through a larger distance, as indeed some define the faster. . . ."
>
> "Since all motion takes place in time, and in each time it is possible for something to be moved and everything movable can be moved faster and also slower, then there can occur in each time the faster and the slower motion. If this is so, then time must be continuous; by the continuous I mean that which is always divisible into divisible parts. . . ."
>
> "If time is continuous, so is distance, for in half the time a thing passes over half the distance, and, in general, in the smaller time the smaller distance, for time and distance have the same divisions; and if one of the two is unlimited, so is the other. . . . For that reason the argument of Zeno assumes an untruth, that one unlimited cannot travel over another unlimited along its own parts, or touch such an unlimited, in a finite time; for length as well as time and, in general, everything continuous, may be considered unlimited in a double sense, namely, according to the [number of] divisions or according to the [distances between the] outermost ends."

This profound criticism directed against Zeno refers evidently to the "Achilles" where we are in danger of modifying, in our minds, the conditions actually existing; we are in danger of thinking of the distance between Achilles and the tortoise as decreasing through an unlimited number of subdivisions, while the time for traversing these successive subdivisions is thought of as about the same for each. There results from this distortion of events an unlimited subdivision of a finite distance, and an unlimited accumulation of finite time-intervals; the former yields what from a modern view-point constitutes a convergent geometric series of space-intervals; the latter a divergent series of time-intervals. A finite distance is made to be traversed in an infinite time.

Aristotle's arguments against the "Achilles" and the "Dichotomy" are the same. He touches upon them in other parts of his *Physics*. Since a line cannot be built up from points, a line cannot actually be subdivided into points.[1]

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[1] See *Physics*, Bk. VI, § 9, quoted earlier in this article.

> "The continued bisection of a quantity is unlimited, so that the unlimited exists potentially, but is actually never reached."[2]
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> "Previously we refuted this [the "Dichotomy"] by the fact that time has unlimitedly many parts, in consequence of which there is no absurdity in the consideration that in unlimited time-intervals one passes over unlimitedly many spaces . . . [but as Aristotle does not consider this a full explanation, he continues:] If one divides a continuous line into halves, he uses one point at two, for he makes it the beginning [of one part] and the end [of the other]; in this manner proceeds he who . . . bisects, but in this division neither the line nor the motion are continuous; for . . . in the continuous there are, to be sure, unlimitedly many halves, but not actually unlimited, only potentially so."[3]

Of the numerous passages in Aristotle's *Physics* which might be quoted as bearing on the "Arrow" we choose the following (Bk. VI, § 8):

> ". . . A thing is at rest, when it is unchanged in one *Now* and still in another *Now*, it itself as well as its parts remaining in the same status. . . . There is no motion nor rest in the *Now* . . . In a time-interval, on the contrary, it [a variable] cannot exist in the same state of rest, for otherwise it would follow that the thing in motion is at rest."

Aristotle's disproof of the "Stade" is given at the end of our first extract from the *Physics* (Bk. VI, § 9), where it is pointed out that Zeno falsely assumes a body to move with the same velocity relative to a body that is at rest, as it would with respect to one in motion.

Further Comments on Aristotle. The main criticism to be made on the profound arguments of Aristotle is that they do not go far enough. The "Dichotomy" and "Achilles" involve the theory of limits, a theory which in very recent time has been found to be in need of reconstruction and which was imperfectly developed in Greek antiquity. Particularly insistent in our mind is the query raised by Zeno's arguments, *how is it possible for a variable to reach its limit?* This query finds no reply in Aristotle. To be noted is also the fact that Aristotle denied the existence of actual infinity, as distinguished from potential infinity. The "Arrow" called for a sharp definition of the *Now* (the instant). Has the instant no duration? Is it, so to speak, a point of time? Or is the instant an infinitesimal—some constant different from zero, yet smaller than any finite quantity? Here Aristotle drew a sharp line; the *Now* was a point of time; it had no duration; in the *Now* there could be neither motion nor rest.

It will be seen that the mathematical concepts herein involved are most fundamental. The concept of a limit involves notions of infinity and is historically connected with the concept of the infinitesimal. All these concepts are basic. It would be difficult to select three other notions more far-reaching in mathematical science than *infinitesimal, infinity, limit.*

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[1] Aristotle, *Physik,* VI, 1, Prantl's ed., p. 281.

[2] Aristotle, *Physik,* III, 7, Prantl's ed., p. 141.

[3] Aristotle, *Physik,* VIII, 8, Prantl's ed., pp. 445, 447. See also Aristotle's *Lib de lineis insecab.,* p. 968.

III.

C. A TWO-THOUSAND YEAR STRUGGLE FOR LIGHT.

1. THE GREEKS AFTER ARISTOTLE.

Whether Plato and Aristotle correctly explained the nature of Zeno's arguments and the purpose which Zeno himself had in mind in presenting them, is of no concern in tracing the history of thought on this subject after the time of Aristotle. Apparently the writings of Plato and Aristotle constituted the sources of information for later writers. Zeno's arguments were given by Aristotle in the form of "fallacies," and it is the influence of these "fallacies" that remains to be traced. There is little doubt that this influence was great upon the development of Greek geometry. Since Aristotle, with all his dialectical skill, was not able to satisfactorily explain all the paradoxes which had arisen in the study of the infinite and of motion, the conclusion of H. Hankel[1] and other recent historians of mathematics is probably correct, that the infinite and infinitesimal were banished from the classic Greek geometry for the sake of greater rigor. We shall see that recently discovered manuscripts of Archimedes confirm this view. Mathematicians assumed that every magnitude is divisible at pleasure. The doctrine of incommensurable lines rests upon the possibility of unlimited divisibility. The denial of the existence of the infinitesimal goes back to Zeno who is reported by Simplicius[2] to have stated: "That which, being added to another, does not make it greater, and being taken away from another does not make it less, is nothing." This momentous question presented itself twenty-two centuries later to Leibniz who gave different answers. In one exposition Leibniz extended the definition of equality so as to declare magnitudes as equal when they differ from one another by an incomparably small quantity. The later Greek mathematicians followed a radical policy toward the infinitesimal; they formally excluded it from demonstrative geometry by a postulate. This was done by Eudoxus (408–355 B. C.), by Euclid (about 300 B. C.) and by Archimedes (287–212 B. C.). Archimedes gives the postulate, which he attributes to Eudoxus, as follows:[3]

> "When two spaces are unequal, it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded."

Euclid in his *Elements* (Bk. V, Def. 4) gives the postulate in the form of a definition:

> "Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other."

The Method of Archimedes, a book formerly thought to be irretrievably lost, but fortunately discovered by Heiberg in 1906 in Constantinople, gives interesting evidence that the notion of infinitesimals, though not used by Archimedes in formal demonstrations, was employed by him in tentative research as a method of discovery. He considered infinitesimals sufficiently scientific to suggest the truths of theorems, but not to furnish rigorous proofs. The process is mechanical, consisting of the weighing of infinitesimal elements, which he calls straight lines or plane areas, but which are really infinitely narrow strips or infinitely thin plane laminæ.[1] The breadth or thickness is regarded as being the same in the elements weighed at any one time.

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[1] H. Hankel, *Geschichte der Mathematik im Alterthum und Mittelalter*, Leipzig, 1874, p. 120.

[2] Simplicius, *Phys.* 30*a*. Quoted by E. Zeller, *History of Greek Philosophy*, Vol. 1, London, 1881, p. 615.

[3] Archimedes, *De quadr. parabol. Praef.*

Further evidence that infinitesimals lingered in the minds of Greek thinkers through the centuries is furnished by the skeptic, Sextus Empiricus (200 A. D.), who advances the paradox that, when a line rotating in a plane about one of its ends describes a circle with each of its points, these concentric circles are of unequal area, yet each circle must be equal to the neighboring circle which it touches.[2] The difficulty encountered here is similar to that raised by Democritus over 500 years earlier.

An interesting remark about Zeno is made by Plutarch (about 90 A. D.) in his life of Pericles. He says that Pericles was also a hearer of Zeno, the Eleatic, who "also perfected himself in an art of his own for refuting and silencing opponents in argument; as Timon of Phlius describes it—

> Also the two-edged tongue of mighty Zeno, who,
> Say what one would, could argue it untrue."

Thus we see that the personality of Zeno and some of the ideas involved in his "paradoxes" appear here and there on the surface of Greek thought, thereby indicating an underground flow of those ideas down the centuries of Greek history.

Sextus Empiricus also gives a version of the "Arrow" much like what we quoted from Diogenes Laertius. He attributes the paradox not to Zeno but to Diodorus Cronus. The existence of motion is disproved thus: If matter moves, it is either in the place in which it is, or the place in which it is not; but it cannot move in the place in which it is, and certainly not in the place in which it is not: hence it cannot move at all. To this Sextus Empiricus replies by stating another argument equally paradoxical and therefore far from illuminating: By the same rule men never die, for if a man die, it must either be at a time when he is alive, or at a time when he is not alive; hence he never dies.

2. The Romans.

The subtleties of Zeno's arguments on motion attracted little attention among the Romans. Lucretius (96–55 B. C.) used the notion of infinity in arguments on atomic theory. He reasoned that one must assume the existence of atoms (indivisible, but not mathematical points), otherwise each body, whether large or small, would consist of an infinite number of parts, and there would be no difference between the largest and the smallest, both being infinity.[1]

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[1] T. L. Heath, *Method of Archimedes*, Cambridge, 1912, p. 8.

[2] Sextus Empiricus, *Adv. math.*, I, III, § 66 ff., ed. Fabricius. p. 322; referred to in K. Lasswitz, *Geschichte der Atomistik* I, Hamburg u. Leipzig, 1890, p. 148.

Cicero and Seneca mention Zeno in passing: the one to display Zeno's love for argument even though faulty, the other to display Zeno's skepticism. Cicero[2] attributes to Zeno the following pointed syllogism: "That which exercises reason is more excellent than that which does not exercise reason; there is nothing more excellent than the universe; therefore the universe exercises reason." Seneca[3] exhibits Zeno as denying not only plurality as did his master, but denying also unity and the real existence of external objects. We have no good reason for accepting either Cicero's or Seneca's view of Zeno's aim and purpose.

3. Medieval Times.

The earliest church father known to interest himself in arguments on motion was St. Augustine (354–430 A. D.). In a dialogue on the question, whether or not the mind of man moves when the body moves, and travels with the body, he is led to a definition of motion, in which he displays some levity. It has been said of scholasticism that it had no sense of humor. Hardly does this apply to St. Augustine in his discussion of the impossibility of motion. He says:

> "When this discourse was concluded, a boy came running from the house to call us to dinner. I then remarked that this boy compels us not only to define motion, but to see it before our very eyes. So let us go, and pass from this place to another; for that is, if I am not mistaken, nothing else than motion."[4]

St. Augustine deserves also the credit of having accepted the existence of the *actually infinite* and to have recognized it as being, not a variable, but a constant. He recognized *all* finite positive integers as an infinity of that type.[5] On this point St. Augustine occupied a radically different and more advanced position than his forerunner, Origen of Alexandria, who took a decided stand against the actually infinite and supported his position by arguments which G. Cantor admits to be the profoundest ever advanced against the actually infinite.[6] In their completest form, these arguments were given many centuries later by the great Italian philosopher of the Middle Ages, Thomas Aquinas (1225(?)–1274).[7] Of importance to us is the nature of the continuum, particularly the linear continuum, as described by Aquinas. It was conceived as *potentially* divisible to infinity, since practically the divisions could not be carried out to infinity. There was, therefore, no minimum line. On the other hand, the point is not a constituent part of a line, since it does not possess the property of infinite divisibility that parts of a line possess, nor can the continuum be constructed out of points. However, a point by its motion has the capacity of generating a line.[1]

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[1] Lucretius, *De rerum natura*, ed. I. Bernays, Leipzig, 1886, I, 615 ff.

[2] *De natura deorum*, Book III, IX.

[3] Epistola 88.

[4] St. Augustine, *De ordine*, II, VI, 18.

[5] S. Augustin, *De civitate Dei*, lib. XII, cap. 19. The chapter is quoted by G. Cantor in "Mitteilungen zur Lehre vom Transfiniten," *Zeitschr. f. Philosophie u. philosoph. Kritik*, Bd. 91, p. 81. Separatabdruck, p. 32.

[6] G. Cantor, *op. cit.*, Separatabdruck, p. 35.

[7] Thomas Aquinas, *Summa theol.*, I, q. 7 a. 4. Quoted by G. Cantor, *op. cit.*, Separatabdruck, p. 36.

This concept of the continuum, as held by Aquinas, is a fair representation of the prevalent medieval scholastic views on this topic. It held a firm ascendancy over the ancient atomistic doctrine which assumed matter to be composed of very small, indivisible particles, possessing the properties of matter itself. No continuum superior to it was created before the nineteenth century.

In his commentaries on Aristotle's *Physics,* Aquinas[2] explains at some length the arguments of Zeno against motion as they are given by Aristotle. Aquinas shows a complete mastery of the subject as expounded by the Stagirite, but hardly presents any new points of view.[3]

Early English Writers. The earliest Englishman known to have written on continuity and infinity is Roger Bacon (1214(?)–1294), the seven-hundredth anniversary of whose birth was celebrated at Oxford in 1914. Bacon argued against the composition of the continuum of *indivisible* parts (different from points), by renewing the arguments presented by the Greeks and the early Arabs. Bacon held that the hypothesis of indivisible parts of uniform size would make the diagonal of a square commensurable with a side; if the ends of an indivisible part of a circle are connected by radii with the center of the circle, then the two radii would intercept an arc on a concentric circle of smaller radius. From this it would follow that the inner circle is of the same length as the outer circle. This is impossible. Bacon argued also against infinity. If time were infinite, it would follow that the part is equal to the whole—a deduction which he considered absurd. Similar arguments lead him to conclude that the world is finite.[4]

The views of Roger Bacon became known more widely through Duns Scotus (1265–1308), the theological and philosophical opponent of Thomas Aquinas. However, Scotus and Aquinas took the same ground in teaching that in the continuum there existed actual, indivisible points. Thereby it is not admitted that the continuum is made up of, or consists wholly of, points; the indivisible points might, for instance, be simply end points. These contentions are directed against the atomists. The arguments are wanting in explicitness and precision. What we said of Aquinas's commentary on Zeno applies also to Duns Scotus.[1] He gives detailed elaborations of Aristotle without offering new explanations of Zeno's puzzles. In place of Achilles and the tortoise he introduces the more familiar travelers, the horse and the ant. His commentaries are annotated by the Franciscan theologian Franciscus de Pitigianis of Arezzo in Italy, who wrote the latter part of the sixteenth century. This annotator expresses himself in favor of the admission of the actual infinity to explain the "Dichotomy" and the "Achilles," but fails to adequately elaborate the subject. Scholastic ideas on infinity and the continuum find expression in the writings of Bradwardine, the English *doctor profundus.* He says that five explanations have been given of the nature of the continuum.[2]

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[1] C. R. Wallner, in *Bibliotheca mathematica,* 3. F., Bd. IV, 1903, pp. 29, 30, gives quotations from Thomas Aquinas, *Opuscula omnia,* 1562, o. 52, p. 369; o. 36, c. 2; o. 44, c. 1; o. 44, c. 2, p. 280.

[2] *Opera omnia,* Tom. II, Pars prima: *Sancti Thomae aqvinatis ex ordine praedicatorum quinti ecclesiae doctoris angelici praeclarissima commentaria in octo Physicorum Aristotelis libros. . . . Ad haec accessit Roberti Linconiensis in eosdem summa.* Parisiis, MDCLX, Lectio XI, pp. 233–237, 352.

[3] A summary of what is given in each of Aristotle's eight books on Physics is given by Robert Grosseteste (1175(?)–1253), bishop of Lincoln, whom Roger Bacon praises as a scientist of high rank. The esteem in which his writings were held appears from the fact that his summary of Aristotle is reproduced four centuries after his death in an edition of Aquinas. Of Zeno's arguments Grosseteste says (page 352): *Ad primum dicitur (sicut prius dictum est id dubitatione præcedenti) scilicet quòd continuum est infinitum secundum potentiam & tale potest transiri. Et sic patet ad secundam rationem. Ad tertium dicitur quòd Zeno dixit ipsum tempus componi ex instantibus, quod non est verum, ideò nec motus nec quies est in instanti, sed in tempore [sicut dixit Philosophus] ideò mobile non est spacio sibi aequali nisi tantum in instanti. Et cum dicitur, aut movetur, aut quiescit, negatur propositio, quia habet veritatem de es quod est in aliquo in quo aptum natum moveri aut quiescere pro tali mensura temporis.*

[4] See *Opera hactenus inedita Baconi,* Fasc. 1, Metaphysica. Edidit Robert Steele. London, p. 11; Jonas Cohn, *Geschichte des Unendlichkeitsproblems,* Leipzig, 1896, pp. 76, 77; K. Lasswitz, *op. cit.,* Vol. I, pp. 193, 195.

GROUPS OF SUBTRACTION AND DIVISION WITH RESPECT TO A MODULUS.

By G. A. MILLER, University of Illinois.

Certain kinds of groups of subtraction and division were explained by the present writer in two articles entitled: "Groups of the fundamental operations of arithmetic" and "Groups of subtraction and division." These articles were published respectively in the *Annals of Mathematics,* volume 6 (1905), page 41; and in the *Quarterly Journal of Mathematics,* volume 37 (1906), page 80. The present article is devoted to more elementary considerations, and has for its main object to exhibit interesting elementary relations between certain groups of subtraction and division, and the corresponding groups of addition and multiplication.

It is well known, and also evident, that the first $m-1$ natural numbers together with zero constitute the cyclic group of order m with respect to addition when the sums are replaced by their least positive residues, or by zero, modulo m. That is, if in the series of numbers

$$0,\ 1,\ 2,\ \cdots,\ m-1$$

each number is replaced by itself increased by α, mod m, where $0 \underset{<}{=} \alpha \underset{<}{=} m-1$, there results a certain substitution on these m numbers, and the totality of the distinct substitutions which can be constructed in this manner constitutes the cyclic group of order m. The order of the substitution corresponding to α is

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[1] Duns Scoti, *Opera Omnia,* T. II: Joannis Duns Scoti Doctoris Subtilis, ordinis minorum, in VIII libros Physicorum Aristotelis Quæstiones, cum annotationibus R. P. F. Francisci Pitigiani arretini, etc. Lvgdvni, MDCXXXIX, Quaestio X, pp. 390–393.

[2] See Maximilian Curtze on the "Tractatus de continuo Bradwardini" in *Zeitschrift f. math. u. Phys.,* XIII Jahrg., Suppl., 1868, Leipzig, p. 88.